"To count, or not to count?"

Tewodros Amdeberhan
Tulane University

Xavier University of Louisiana
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Dialog through times

- "What's Combinatorics?" - Socrates (400 BC)
- "To count, or not to count." - Shakespeare (1600's AD)
- " Uses a bijection. " - Combinatorialist (today)
- "Huh?" - Anonymous.

WHO DO YOU CALL?

$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}\binom{n}{k} = (-1)^n.$$

$$\sum_{k=0}^{n}\binom{n+k}{k}\binom{n}{k} = ?$$

$$\sum_{k}(-1)^k\binom{n+a}{n+k}\binom{n+b}{b+k}\binom{a+b}{a+k} = \binom{n+b+a}{n,b,a}.$$

$$\sum_{n=1}^{\infty}\frac{\binom{2n-1}{n}^2}{(n+1)\cdot 16^n} = \frac{1}{\pi}-\frac{1}{4}.$$

$$2\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2} = 3\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}n^2}.$$

W=Herbert Wilf

Z=Doron Zeilberger

The Wilf-Zeilberger theory

The Joy of Operator Notation

Let N be the shift operator in n: $N\,f(n) = f(n+1)$.

(Warm-up a lá Fibonacci).

$$1, 1, 2, 3, 5, 8, 13, \ldots \qquad \Longleftrightarrow \qquad F_{n+2} = F_{n+1} + F_n.$$

$$\Longleftrightarrow \qquad (N^2 - N - 1)F_n = 0.$$

Example. Prove that $F_{n+4} = F_{n+2} + 2F_{n+1} + F_n$.

Proof. Convert to $(N^4 - N^2 - 2N - 1)F_n = 0$. Even better

$$(N^2 + N + 1)(N^2 - N - 1)F_n = 0. \qquad \text{QED.}$$

A hypergeometric term $F(n, k)$?

$$\frac{F(n+1, k)}{F(n, k)}$$

and

$$\frac{F(n, k+1)}{F(n, k)}$$

are rational functions of n and k.

Hypergeometric + Operators

$$N\,F(n,k) = F(n+1,k) \qquad \text{and} \qquad K\,F(n,k) = F(n,k+1).$$

Example. $F(n,k) = \binom{n}{k}$ is a hypergeometric term. Why?

$$\frac{N\,F(n,k)}{F(n,k)} = \frac{n+1}{n-k+1} \qquad \text{and} \qquad \frac{K\,F(n,k)}{F(n,k)} = \frac{n-k}{k+1}.$$

Rewrite these

$$[-Nk + (n+1)N - (n+1)]F = 0 \implies P \cdot F = 0$$
$$[(K+1)k - n]F = 0 \implies Q \cdot F = 0.$$

The ansatz in $\mathbb{Q}[n, k, N, K]$

Choose A and B such that

$$A \cdot P + B \cdot Q$$

is **independent** of k and that

$$[A \cdot P + B \cdot Q]F = 0.$$

. . . example continued

$$[-Nk + (n+1)N - (n+1)]F = 0 \implies P \cdot F = 0$$
$$[(K+1)k - n]F = 0 \implies Q \cdot F = 0.$$

Eliminating k:

$$(K+1)(i) + N(ii) = 0$$

results in

$$(n+1)[N\,K - K - 1]F = 0.$$

Rewrite

$$N\,K - K - 1 = (N-2) + (K-1)(N-1)$$

so that

$$[(n+1)(N-2) + (K-1)(n+1)(N-1)]F = 0.$$

. . . example continued

Repeated here

$$(N-2)F + (K-1)(N-1)F = 0.$$

Let $G(n,k) = -(N-1)F(n,k)$.

Get

$$(N-2)F(n,k) = (K-1)G(n,k)$$

where

$$F(n,k) = \binom{n}{k} \qquad \text{and} \qquad G(n,k) = -\binom{n}{k-1}.$$

Wrap it up: $S_{n+1} = 2S_n$

Repeated here

$$(N-2)F(n,k) = (K-1)G(n,k)$$

reads as

$$\binom{n+1}{k} - 2\binom{n}{k} = \binom{n}{k-1} - \binom{n}{k}$$

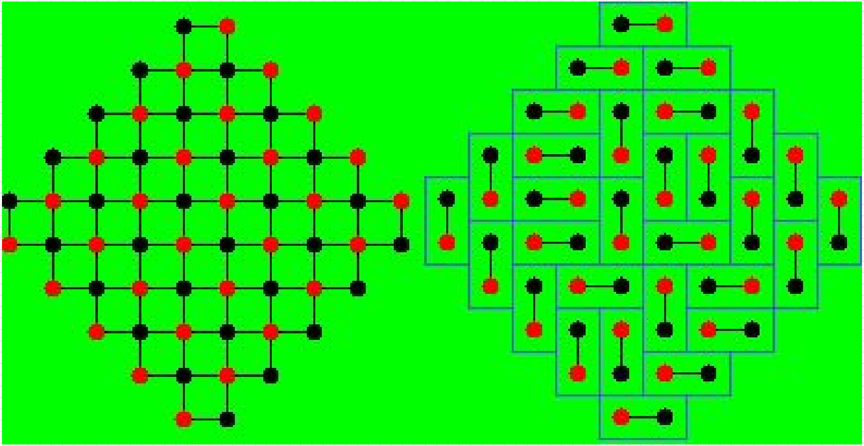
leading to

$$\sum_k \binom{n+1}{k} - 2\sum_k \binom{n}{k} = 0.$$

That means

$$\sum_k \binom{n}{k} = 2^n. \qquad \text{QED}$$

Dimers in Aztec Diamond



Determinants

$$\det [A_{ij}]_{i,j}^{1,n} = \sum_{\sigma \in \mathfrak{S}_{\mathbf{n}}} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} A_{k,\sigma(k)}.$$

Examples are not just a way teach, they are the only way to teach.

- A. Einstein.

Vandermonde $V_3(x, y, z)$

$$det \begin{pmatrix} x^0 & y^0 & z^0 \\ x^1 & y^1 & z^1 \\ x^2 & y^2 & z^2 \end{pmatrix} = (y - x)(z - x)(z - y) = \prod_{a<b} (b - a).$$

Calculus 1: pop quiz

$$\int_0^1 \mathbf{x}^{\mathbf{n-1}}\,dx = \frac{\mathbf{1}}{\mathbf{n}}.$$

Example: The Hilbert Matrix

$$\det\begin{pmatrix} 1/1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} = \det\begin{pmatrix} \int_0^1 x^0 & \int_0^1 x^1 & \int_0^1 x^2 \\ \int_0^1 x^1 & \int_0^1 x^2 & \int_0^1 x^3 \\ \int_0^1 x^2 & \int_0^1 x^3 & \int_0^1 x^4 \end{pmatrix}$$

$$= \det\begin{pmatrix} \int_0^1 x^0 & \int_0^1 \mathbf{y}^{\mathbf{1}} & \int_0^1 \mathbf{z}^{\mathbf{2}} \\ \int_0^1 x^1 & \int_0^1 \mathbf{y}^{\mathbf{2}} & \int_0^1 \mathbf{z}^{\mathbf{3}} \\ \int_0^1 x^2 & \int_0^1 \mathbf{y}^{\mathbf{3}} & \int_0^1 \mathbf{z}^{\mathbf{4}} \end{pmatrix}$$

$$= \int_0^1 \int_0^1 \int_0^1 \det\begin{pmatrix} x^0 & \mathbf{y}^{\mathbf{1}} & \mathbf{z}^{\mathbf{2}} \\ x^1 & \mathbf{y}^{\mathbf{2}} & \mathbf{z}^{\mathbf{3}} \\ x^2 & \mathbf{y}^{\mathbf{3}} & \mathbf{z}^{\mathbf{4}} \end{pmatrix}$$

$$= \int_0^1 \int_0^1 \int_0^1 x^0 y^1 z^2 \det\begin{pmatrix} x^0 & \mathbf{y}^{\mathbf{0}} & \mathbf{z}^{\mathbf{0}} \\ x^1 & \mathbf{y}^{\mathbf{1}} & \mathbf{z}^{\mathbf{1}} \\ x^2 & \mathbf{y}^{\mathbf{2}} & \mathbf{z}^{\mathbf{2}} \end{pmatrix}$$

Vandermonde $V(x, y, z)$

$$= [x^0 y^1 z^2 - y^0 x^1 z^2 + y^0 z^1 x^2 - z^0 y^1 x^2 + z^0 x^1 y^2 - x^0 z^1 y^2]$$

$$\begin{aligned} \det(\text{Hilbert}) &= \frac{1}{3!} \iiint V^2(x, y, z)\, dxdydz \\ &= \frac{1}{3!} \iiint (z-x)^2 (z-y)^2 (y-x)^2 dxdydz. \end{aligned}$$

More generally

$$\det \left[\frac{1}{i+j-1} \right]_{i,j}^{1,n} = \frac{1}{n!} \int_{\square^n} d\mathbf{x} \prod_{j>i} (x_j - x_i)^2.$$

Enter Charles L. Dodgson (a.k.a Lewis Carroll)

Alice in Wonderland

$\Rightarrow$

Determinants in Wonderland

Dodgson condensation

$$\det \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} =$$

$$\frac{\det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \det \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} - \det \begin{pmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix} \det \begin{pmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{pmatrix}}{\det \left(A_{22} \right)}$$

Three-term non-linear recursive formula

$$\det M_n = \frac{NW_{n-1} \cdot SE_{n-1} - SW_{n-1} \cdot NE_{n-1}}{CTR_{n-2}}.$$

EXAMPLE: $\det[\binom{i+j}{i}]$

Generalize

$$\det\left[\binom{i+j+a+b}{i+a}\right]_{i,j=0}^{c-1} = \prod_{k=0}^{c-1}\prod_{j=0}^{b-1}\prod_{i=0}^{a-1}\frac{i+j+k+2}{i+j+k+1}.$$

Number of plane partitions in a box!

Hey, what happened to ...

... the determinant of the Hilbert matrix

$$\det \left[\frac{1}{i+j-1} \right]_{i,j} ?$$

Generalize to trivialize!

$$\det\left[\frac{1}{\alpha_i+\beta_j}\right]_{i,j}^{1,n} = \frac{\prod_{i<j}(\alpha_j-\alpha_i)(\beta_i-\beta_j)}{\prod_{i,j}(\alpha_j-\beta_i)}.$$

Choose $\alpha_i = i$, $\beta_j = j-1$. Get

$$\det\left[\frac{1}{i+j-1}\right]_{i,j}^{1,n} = \frac{1}{n!\prod_{k=0}^{n-1}\binom{2k}{k}\binom{2k+1}{k}}.$$

BONUS:

$$\int_{\square^n} d\mathbf{x}_n \prod_{j>i}(x_j-x_i)^2 = \frac{1}{\prod_{k=0}^{n-1}\binom{2k}{k}\binom{2k+1}{k}}.$$

ZZZZZzzzzzzzzzzz. . .

Are you tired

of

counting??

J. Serrin; H. Weinberger

Suppose

$$\begin{cases} \Delta u & = -1 \qquad \Omega \\ u & = 0 \qquad \partial\Omega \\ \frac{\partial u}{\partial \nu} & = -cr \qquad \partial\Omega. \end{cases}$$

- $0 \in \Omega \subset \mathbb{R}^N$ bounded, simply connected.
- $\partial\Omega$ smooth enough.
- A solution $u(x)$ exists.

Then what??

SQUEEZE & POUR!

Key: $\nu = \pm \frac{\nabla u}{\|\nabla u\|}$

This is the goal

$$\begin{cases} \Phi|_{\partial\Omega} = 0 \\ \Delta\Phi \geq 0 \\ \int_\Omega \Phi = 0 \end{cases} \qquad \Longrightarrow \Phi|_\Omega \equiv 0.$$

Proof Outline

$$\text{Let } \Phi = \nabla u \cdot \nabla u - c^2 r^2.$$

$$\Phi|_{\partial\Omega} = \left(\frac{\partial u}{\partial \nu}\right)^2 - c^2 r^2 = 0.$$

$$\Delta\Phi = 2\sum_{i,j}\left(u_{ij} + \frac{\delta_{ij}}{N}\right)^2 + \frac{2}{N}(1 - c^2 N^2) \geq 0.$$

$$\int_\Omega \Phi dx = \int_{\partial\Omega} u \frac{\partial u}{\partial \nu} - \int_\Omega u \Delta u - c^2 \int_\Omega r^2 dx$$

$$= \int_\Omega u dx - c^2 \int_\Omega r^2 dx$$

$$\underbrace{= 0.}_{\textit{wishful}}$$

Let $h = (2+N)u - div(xu) = 2u - \sum_i x_i u_i \Rightarrow h$ is *harmonic.*

$$\int_\Omega h\Delta u - u\Delta h = \int_{\partial\Omega} h\frac{\partial u}{\partial \nu} - u\frac{\partial h}{\partial \nu} \Longrightarrow \int_\Omega h = c\int_{\partial\Omega} rh.$$

$$\int_\Omega h = (2+N)\int_\Omega u.$$

$$c\int_{\partial\Omega} rh = -c\int_{\partial\Omega} r^2\frac{\partial u}{\partial r} = -c\int_{\partial\Omega} r^2 \underbrace{\frac{\partial u}{\partial \nu}\frac{\partial r}{\partial \nu}}_{\text{not chain rule}}$$

$$= \frac{c^2}{4}\int_{\partial\Omega}\frac{\partial r^4}{\partial \nu} = \frac{c^2}{4}\int_\Omega \Delta(r^4)$$

$$= c^2(N+2)\int_\Omega r^2.$$

" $\geq$ " into " $=$ "

$$\begin{cases} \Phi|_{\partial\Omega} = 0 \\ \Delta\Phi \geq 0 \\ \int_\Omega \Phi = 0 \end{cases} \implies \Phi|_\Omega \equiv 0.$$

$$\Delta\Phi = 2\sum_{i,j}\left(u_{ij} + \frac{\delta_{ij}}{N}\right)^2 + \frac{2}{N}(1 - c^2N^2) \equiv 0.$$

$$\implies u_{ij} + \frac{\delta_{ij}}{N} = 0$$

$$\implies u = a - \frac{r^2}{2N}$$

$$\implies r \equiv \textbf{constant}.$$

We've just proven

Theorem (T.A.) *Suppose* $\Omega \subset \mathbb{R}^N$ *is bounded, simply connected with smooth boundary,* $r^2 = x_1^2 + \cdots + r_N^2$ *and* $c > 0$ *a constant. If there exists a solution* $u \in C^2(\bar{\Omega})$ *to*

$$\begin{cases} \Delta u = -1 & \Omega \\ u = 0 & \partial\Omega \\ \frac{\partial u}{\partial \nu} = -cr & \partial\Omega, \end{cases}$$

then (a) Ω *is an N-dimensional sphere, (b) the solution is radial.*


That's all Folks!