# Wallis and Landen:
# A tale of two integrals
# Xavier University Colloquium

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# Wallis' formula: The first formulation

$$W_m = \int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m}$$

$$\frac{1}{\pi}\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{1}{2^{2m+1}}\binom{2m}{m}$$

Analysis = Discrete Mathematics

# The first recurrence

Integrate by parts:

$$\frac{1}{(x^2+1)^{m+1}} = \frac{x}{2}\frac{d}{dx}\left(-\frac{1}{m+1}\frac{1}{(x^2+1)^{m+1}}\right) + \frac{1}{(x^2+1)^{m+2}}$$

Integration gives

$$W_{m+1} = \frac{2m+1}{2(m+1)} W_m$$

# A nice modification

$$W_{m+1} = \frac{2m+1}{2(m+1)} W_m$$

Experimentally one can guess

$$W_m = \frac{\pi}{2^{2m+1}} \binom{2m}{m}$$

now define

$$Y_m = \frac{2^{2m+1}}{\pi \binom{2m}{m}} W_m$$

The recurrence for $Y_m$ is

$$Y_{m+1} = Y_m$$

# Guessing the formula

## Some data

$$\int_0^\infty \frac{dx}{(1+x^2)^{31}} = \frac{739153634780383 9\pi}{144115188075855872}$$

Denominator is $2^{57}$

Many other examples suggest to multiply the integral by $2^{2m+1}$

Define

$$T_m = \frac{2^{2m+1}}{\pi} \int_0^\infty \frac{dx}{(x^2+1)^{m+1}}$$

$$T_{30} = 118264581564861424$$

and this factors as

$$T_{30} = 2^4 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59$$

Experiments show that $T_m$ is an integer and has no prime factors larger than $2m$.

# Guessing the formula

This suggests the construction of $S_m = \frac{(2m)!}{T_m}$

$$S_{30} = 703590796385458823746892467806561195760321617199104000000000000000$$

This factors as

$$S_{30} = 2^{52} \cdot 3^{28} \cdot 5^{14} \cdot 7^{8} \cdot 11^{4} \cdot 13^{4} \cdot 17^{2} \cdot 19^{2} \cdot 23^{2} \cdot 29^{2}$$

Many other examples show that $S_m$ has no primes higher than $m$ and they all appear with even power

This motivates the definition of $U_m = \frac{S_m}{m!^2} = \frac{(2m)!}{m!^2} \times \frac{1}{T_m}$

$$T_m = \frac{2^{2m+1}}{\pi} \int_0^\infty \frac{dx}{(x^2+1)^{m+1}}$$

# Guessing the formula

Then

$$U_{30} = 1$$

Many other examples give the same answer: $U_m = 1$.

This gives the guess

$$\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m}$$

Before you prove a formula, you have to have a formula

# Wallis' formula. Continuation.

John Wallis

`Savillian professor at Oxford`

Editor of Newton's work.

Arithmetica Infinitorum, 1656

Best contribution to Mathematics

$\infty$

# Iterate the recurrence

$$W_{m+1} = \frac{2m+1}{2(m+1)} W_m$$

The infinite product becomes

$$\prod_{n=1}^{\infty} \frac{2n}{2n-1} \, \frac{2n}{2n+1} = \frac{\pi}{2}$$

This is before Infinitesimal Calculus.

The infinite product is equivalent to the integral we started with.

# A complicated change of variables

$$W_m = \int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m}$$

$$x = \tan t$$

produces

$$W_m = \int_0^{\pi/2} \cos^{2m} t \, dt$$

# A new proof of Wallis' formula. More complicated

$$W_m = \int_0^{\pi/2} \cos^{2m} t \, dt$$

$$\begin{aligned} W_m &= \frac{1}{2^m} \int_0^{\pi/2} (1 + \cos 2t)^m \, dt \\ &= \frac{1}{2^{m+1}} \int_0^{\pi} (1 + \cos x)^m \, dx \\ &= \frac{1}{2^{m+1}} \sum_{j=0}^{m} \binom{m}{j} \int_0^{\pi} \cos^j x \, dx \end{aligned}$$

by symmetry integrals with $j$ odd vanish

$$W_m = \frac{1}{2^m} \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k} W_k$$

# Continuation

make the same guess as before
Wallis' identity is equivalent to

$$\sum_k 2^{-2k}\binom{m}{2k}\binom{2k}{k} = 2^{-m}\binom{2m}{m}$$

**Proof**

$$ct(binomial(m,2i)binomial(2i,i)2^{-2i},1,i,m,N)$$

$ct$ is **creative telescoping** due to H. Wilf and D. Zeilberger
better

$$ct(2^m binomial(2m,m)^{-1}binomial(m,2i)binomial(2i,i)2^{-2i},1,i,m,N)$$

gives

$$f(m+1) = f(m)$$

# The indefinite integral

**Theorem**

*The integral*

$$W_n(x) = \int_0^x \frac{dt}{(1+t^2)^{n+1}}$$

*is given by*

$$W_n(x) = \frac{\binom{2n}{n}}{2^{2n}} \left( \text{arctan } x + \sum_{k=1}^{n} \frac{2^{2k}}{2k\binom{2k}{k}} \frac{x}{(1+x^2)^k} \right).$$

This is a finite ${}_2F_1$.

# Doubling the angle

• $\cot\theta$ is the natural rational function
• $\cot$ is the an elliptic function with
one period becoming infinite

$$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$$

$$y = \frac{x^2-1}{2x}$$

Make the change of variables $y = (x^2-1)/2x$ in the integral

$$\int_{\mathbb{R}} f(x)\,dx$$

# Doubling the angle. Continuation

$$y = \frac{x^2-1}{2x}$$

$$f_{\pm}(y) = f(y+\sqrt{y^2+1}) \pm f(y-\sqrt{y^2+1})$$

$$\mathfrak{L}(f)(y) = f_{+}(y) + \frac{y\, f_{-}(y)}{\sqrt{y^2+1}}$$

$$\int_{\mathbb{R}} f(x)\, dx = \int_{\mathbb{R}} \mathfrak{L}(f)(y)\, dy$$

# Doubling the angle. Continuation

$$\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} \mathfrak{L}(f)(y)\,dy$$

**Theorem**

*If $f$ is a rational function of $x$, then so is $\mathfrak{L}(f)(y)$.*

$\mathfrak{L}$ is called the **rational Landen map**

This leads to interesting dynamical questions.
$f(x) = 1/(x^2+1)$ is a fixed point.

All of them are classified.

# A warm-up example

## Example

Apply the transformation to obtain

$$\frac{1}{ax^2+b} \mapsto \frac{2(a+b)}{4abx^2+(a+b)^2}$$

This gives a map from $\mathbb{R}^2$ to itself

$$\{a,b\} \mapsto \left\{ \frac{2ab}{a+b}, \frac{a+b}{2} \right\}$$

# A warm-up example. Continuation

## Example

This gives a map from $\mathbb{R}^2$ to itself

$$\{a,b\} \mapsto \left\{ \frac{2ab}{a+b}, \frac{a+b}{2} \right\}$$

The second component is the `arithmetic mean` of $a$ and $b$

$$AM(a,b) = \frac{1}{2}(a+b).$$

The first component is the `harmonic mean`

$$\frac{1}{\texttt{Harm}(a,b)} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right).$$

# Doubling the angle. Continuation

## Example

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = \int_{-\infty}^{\infty} \frac{dx}{a_1x^2+b_1x+c_1}$$

$$a_1 = \frac{(c+a)^2-b^2}{2(c+a)}, \quad b_1 = \frac{b(c-a)}{c+a}, \quad c_1 = \frac{2ca}{c+a}$$

Iterate this map: $a_n \to L$, $b_n \to 0$, $c_n \to L$

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = L \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi \times L$$

# The quartics

## Theorem

*The integral*

$$I_4(a_0, a_2; b_0, b_2, b_4) = \int_{-\infty}^{\infty} \frac{a_0 + a_2 x^2}{b_0 + b_2 x^2 + b_4^4} \, dx$$

*is invariant under the change of parameters*

$$\begin{aligned}
a_0 &\mapsto 2(a_0 + a_2)(b_0 + b_2 + b_4) \\
a_2 &\mapsto 8(a_2 b_0 + a_0 b_4) \\
b_0 &\mapsto (b_0 + b_2 + b_4)^2 \\
b_2 &\mapsto 4(b_0 b_2 + 4 b_0 b_4 + b_2 b_4) \\
b_4 &\mapsto 16 b_0 b_4.
\end{aligned}$$

# Quartic: continuation

## Corollary

*The normalization of the denominator coefficients leads to the planar system*

$$\begin{aligned} b_2 &\mapsto \frac{4(b_2 + 4b_4 + b_2 b_4)}{1 + b_2 + b_4^2} \\ b_4 &\mapsto \frac{16 b_4}{(1 + b_2 + b_4)^2} \end{aligned}$$

# The elliptic context

Change harmonic by geometric:

$$a_{n+1} = \frac{a_n + b_n}{2} \quad b_{n+1} = \sqrt{a_n b_n}$$

$$a_n,\, b_n \to \text{ common limit } = AGM(a,b)$$

$$AGM(a,b) = AGM\left(\frac{a+b}{2}, \sqrt{ab}\right)$$

This is due to `John Landen` $(1719 - 1790)$.

## Theorem

*The arithmetic-geometric mean is given by*

$$\frac{1}{AGM(1+k, 1-k)} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

# The elliptic context. Continuation

$$G(a,b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}$$

$$G(a,b) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{\sqrt{(a^2+x^2)(b^2+x^2)}}$$

$$G(a,b) = G\left(\frac{a+b}{2}, \sqrt{ab}\right)$$

D. Newman's proof (1985) $x \mapsto x + \sqrt{x^2+ab}$

He told me about it. I ignored him

Looked like Calculus to me.

Message: Listen to the elders

# The limit

$$G(a,b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}$$

$$G(a,b) = G(a_n, b_n)$$

passing to the limit

$$G(a,b) = \frac{\pi}{2\,AGM(a,b)}$$

Three quantities $G(a,b)$, $AGM(a,b)$, $\pi$
Two easy to compute
Now you can compute $\pi$ to trillion of digits

# Degree six Landen

It turns out that for rational functions of degree $6$

$$\frac{\text{something}}{x^6 + ax^4 + bx^2 + 1} \rightarrow \frac{\text{something}}{x^6 + a_1x^4 + b_1x^2 + 1}$$

where

$$a_1 = \frac{ab + 5a + 5b + 9}{(a + b + 2)^{4/3}} \qquad b_1 = \frac{a + b + 6}{(a + b + 2)^{2/3}}$$

Now iterate (show landen-pic.nb)

# Open problems

- Develop Landen transformations for

$$\int_0^\infty \frac{dx}{ax^2 + bx + c}$$

with $b \neq 0$.

If you are interested in nice problems
send me an email
I have lots of them

# THANKS FOR YOUR ATTENTION